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Approximately optimal tracking control for discrete time-delay systems with disturbances

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Abstract

Optimal tracking control (OTC) for discrete time-delay systems affected by persistent disturbances with quadratic performance index is considered. By introducing a sensitivity parameter, the original OTC problem is transformed into a series of two-point boundary value (TPBV) problems without time-advance or time-delay terms. The obtained OTC law consists of analytic feedforward and feedback terms and a compensation term which is the sum of an infinite series of adjoint vectors. The analytic feedforward and feedback terms can be found by solving a Riccati matrix equation and two Stein matrix equations. The compensation term can be obtained by using an iteration formula of the adjoint vectors. Observers are constructed to make the approximate OTC law physically realizable. A simulation example shows that the approximate approach is effective in tracking the reference input and robust with respect to exogenous persistent disturbances.

Keywords: Discrete systems; Time-delay systems; Disturbances; Optimal control; Tracking control

1. Introduction

Time-delay is quite common in practical control systems. The optimal control problems for time-delay have been deeply investigated, and many results on optimal control problems for input-delay and state-delay systems have been obtained [1–3]. The optimal control problem for time-delay systems with a quadratic performance index usually leads to a TPBV problem with both time-advance and time-delay terms, which is very difficult to be solved precisely [4–7]. Although discrete time-delay systems can be transformed into systems without time-delay by expanding the system's dimension, systems of high dimension will suffer from a "dimension disaster", which can burden computers with geometric progression. Therefore, finding an approximate optimal control law is one of the important aims of researchers. A successive approximation approach has been presented to generate the suboptimal control law of time-delay or nonlinear systems, which avoids the complexity of solving the TPBV problems with time-delay or nonlinear terms [4,7].

There are various external disturbances in practice, which affect the performance of systems. Therefore, it is worthwhile to study the optimal control of systems with exogenous persistent disturbances. Recently, there are many studies on optimal control problems, which are relevant to disturbance rejection [8–10].

The aims of this paper are to address the OTC problem using an infinite quadratic cost functional for a discrete system with state time-delay, which is affected by persistent disturbances. By introducing a sensitivity parameter ε into the variables of the systems and expanding the Maclaurin

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series around it at $\varepsilon = 0$, we can transform the original TPBV problem generated in the OTC control problems with an infinite horizon quadratic cost functional into a series of TPBV problems without delay or advance terms accordingly. By solving the TPBV problem sequence recursively, we can obtain the OTC law consisting of linear analytic state feedback terms and a compensation term. The linear analytic state feedback terms can be uniquely solved by a Riccati matrix equation and two Stein matrix equations. The compensation term can be obtained with a recursion formula of adjoint vectors. The existence and uniqueness of the OTC law are proved and the physically realizable problem of the OTC law is considered. Finally, performance of the obtained OTC law is verified by an illustrative example. The simulation results show that the obtained regulator is easy to implement and robust with respect to additive persistent disturbances.

2. Problem statement

Consider discrete systems with time-delay described by

$$\begin{aligned} &\boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{A}_1\boldsymbol{x}(k-\tau) + \boldsymbol{B}\boldsymbol{u}(k) + \boldsymbol{M}\boldsymbol{w}(k) \\ &\boldsymbol{x}(k) = \boldsymbol{\varphi}(k), \quad k \in I_0 \\ &\boldsymbol{y}(k) = \boldsymbol{C}\boldsymbol{x}(k) \end{aligned} \tag{1}$$

where $\boldsymbol{x} \in \boldsymbol{R}^n$, $\boldsymbol{u} \in R^r$, $\boldsymbol{y} \in \boldsymbol{R}^m$ are the state, control input, and output vectors, respectively; τ is a positive integer time-delay; $\boldsymbol{\varphi}(k)$ is the known initial state function vector; $\boldsymbol{A}$, $\boldsymbol{A}_1$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{M}$ are constant matrices of appropriate dimensions; $I_0 = \{-\tau, -\tau+1, \ldots, 0\}$; $\boldsymbol{w}(k) \in \boldsymbol{R}^m$ is a disturbance vector which can be given by the following exosystem:

$$\begin{aligned} &\boldsymbol{v}(k+1) = \boldsymbol{G}\boldsymbol{v}(k) \\ &\boldsymbol{w}(k) = \boldsymbol{D}\boldsymbol{v}(k) \end{aligned} \tag{2}$$

where $\boldsymbol{v} \in \boldsymbol{R}^n$; $\boldsymbol{G}$ and $\boldsymbol{D}$ are constant matrices of appropriate dimensions.

Remark 1. The poles of system (2) satisfy $|\lambda_i(\boldsymbol{G})| \leqslant 1$ $(i = 1,2, \cdots ,n)$; in addition, its poles on the unit circle must be the simple root of minimum polynomial of G in order to guarantee that exosystem (2) is stable or asymptotically stable.

The reference input $\boldsymbol{y}_s$ which is tracked by $\boldsymbol{y}$ in system (1) is given by

$$\begin{aligned} &\boldsymbol{z}(k+1) = \boldsymbol{F}\boldsymbol{z}(k) \\ &\boldsymbol{y}_s(k) = \boldsymbol{H}\boldsymbol{z}(k) \end{aligned} \tag{3}$$

where $\boldsymbol{z} \in \boldsymbol{R}^n$, $\boldsymbol{y}_s \in \boldsymbol{R}^m$; $\boldsymbol{F}$ and $\boldsymbol{H}$ are constant matrices of appropriate dimensions. Assume that exosystem (3) is asymptotically stable. The infinite horizon quadratic performance index is described by

$$\boldsymbol{J} = \sum_{k=0}^{\infty} \left[\boldsymbol{e}^{\mathrm{T}}(k)\boldsymbol{Q}\boldsymbol{e}(k) + \boldsymbol{u}^{\mathrm{T}}(k)\boldsymbol{R}\boldsymbol{u}(k)\right] \tag{4}$$

where $\boldsymbol{Q} \in \boldsymbol{R}^{n\times n}$, $\boldsymbol{R} \in \boldsymbol{R}^{r\times r}$ are positive-definite matrices; $\boldsymbol{e}(k)$ is the output error given by

$$\boldsymbol{e}(k) = \boldsymbol{y}_s(k) - y(k) \tag{5}$$

Note that system (1) is affected by the persistent disturbances $\boldsymbol{v}(k)$ in exosystem (2), which is stable but may not be asymptotically stable. $\boldsymbol{e}(k)$ may not approximate to zero as k increases to infinity. Therefore, the infinite horizon quadratic performance index may not be convergent due to persistent disturbances in system (1). We can choose the infinite time quadratic performance index to be

$$\boldsymbol{J} = \lim_{N\to\infty} \frac{1}{\boldsymbol{N}} \sum_{k=0}^{N} \left[\boldsymbol{e}^{\mathrm{T}}(k)\boldsymbol{Q}\boldsymbol{e}(k) + \boldsymbol{u}^{\mathrm{T}}(k)\boldsymbol{R}\boldsymbol{u}(k)\right] \tag{6}$$

The OTC problem is to find an optimal control law $\boldsymbol{u}^*(k)$, which minimizes quadratic performance index (4) or (6) subject to the difference equations constraint (1).

3. Design of the optimal tracking controller

3.1. TPBV problem

We can obtain an infinite TPBV problem by applying the necessary conditions of the maximum principle to problems (1)–(4) in the form of

$$\begin{aligned} &\boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{A}_1\boldsymbol{x}(k-\tau) - \boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{\lambda}(k+1) + \boldsymbol{M}\boldsymbol{w}(k) \\ &\boldsymbol{\lambda}(k) = \boldsymbol{C}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{C}\boldsymbol{x}(k) - \boldsymbol{C}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{H}\boldsymbol{z}(k) + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\lambda}(k+1) + \boldsymbol{A}_1^{\mathrm{T}}\lambda(k+\tau+1) \\ &\boldsymbol{x}(k) = \boldsymbol{\phi}(k), \ k \in I_0 \\ &\boldsymbol{\lambda}(\infty) = 0 \end{aligned} \tag{7}$$

The OTC law can be expressed as

$$\boldsymbol{u}(k) = -\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{\lambda}(k+1) \tag{8}$$

Unfortunately, since the TPBV problem in (7) contains both time-delay and time-advance terms, it is extremely difficult to obtain its precise analytical solution. Therefore, it is necessary to find an approximation approach that can be used to solve this TPBV problem.

3.2. Simplification of the original TPBV problem

To simplify the original TPBV problem, we introduce a sensitivity parameter ε and construct a new TPBV problem with ε as follows:

$$\begin{aligned} \boldsymbol{x}(k+1,\varepsilon) &= \boldsymbol{A}\boldsymbol{x}(k,\varepsilon) + \varepsilon\boldsymbol{A}_1\boldsymbol{x}(k-\tau,\varepsilon) \\ &\quad - \boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{\lambda}(k+1,\varepsilon) + \boldsymbol{M}\boldsymbol{w}(k) \\ \boldsymbol{\lambda}(k,\varepsilon) &= \boldsymbol{C}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{C}\boldsymbol{x}(k,\varepsilon) - \boldsymbol{C}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{H}\boldsymbol{z}(k) \\ &\quad + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\lambda}(k+1,\varepsilon) + \varepsilon\boldsymbol{A}_1^{\mathrm{T}}\lambda(k+\tau+1,\varepsilon) \\ \boldsymbol{x}(k,\varepsilon) &= \boldsymbol{\varphi}(k), \quad k \in I_0 \\ \boldsymbol{\lambda}(\infty,\varepsilon) &= 0 \end{aligned} \tag{9}$$

We also construct a new control law in the form of

$$\boldsymbol{u}(k,\varepsilon) = -\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{\lambda}(k+1,\varepsilon) \tag{10}$$

Assume that $\boldsymbol{x}(k,\varepsilon)$, $\boldsymbol{\lambda}(k,\varepsilon)$ and $\boldsymbol{u}(k,\varepsilon)$ are differentiable infinitely at $\varepsilon = 0$ and that their Maclaurin series can be described by

$$\begin{aligned}
&\boldsymbol{u}(k,\varepsilon) = \sum_{i=0}^{\infty}\frac{\varepsilon^i}{i!}\boldsymbol{u}^{(i)}(k),\\
&\boldsymbol{x}(k,\varepsilon) = \sum_{i=0}^{\infty}\frac{\varepsilon^i}{i!}\boldsymbol{x}^{(i)}(k), \quad \boldsymbol{\lambda}(k,\varepsilon) = \sum_{i=0}^{\infty}\frac{\varepsilon^i}{i!}\boldsymbol{\lambda}^{(i)}(k)
\end{aligned} \tag{11}$$

where $(\bullet)^{(i)} = \partial^i(\bullet)/\partial\varepsilon^i|_{\varepsilon=0}$. Obviously, if the series in (11) are convergent at $\varepsilon = 1$, new TPBV problem (9) is equivalent to original TPBV problem (7) when $\varepsilon = 1$. The OTC law can be written in the form of

$$\boldsymbol{u}^*(k) = \boldsymbol{u}(k,1) = \sum_{i=0}^{\infty}\frac{1}{i!}\boldsymbol{u}^{(i)}(k). \tag{12}$$

In the following discussion, we assume that the series in (11) is convergent at $\varepsilon = 1$. It is almost impossible to prove the conditions satisfying this assumption but we can validate the convergence of the series in (11) at $\varepsilon = 1$ by using example simulations. Fortunately, we have not encountered any case where this series diverges.

Substituting (11) into (9) and comparing the coefficients of the same order terms with respect to ε, we obtain

$$\begin{aligned}
&\boldsymbol{x}^{(0)}(k+1) = \boldsymbol{A}\boldsymbol{x}^{(0)}(k) - \boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{\lambda}^{(0)}(k+1) + \boldsymbol{M}\boldsymbol{w}(k)\\
&\boldsymbol{\lambda}^{(0)}(k) = \boldsymbol{C}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{C}\boldsymbol{x}^{(0)}(k) - \boldsymbol{C}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{H}\boldsymbol{z}(k) + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\lambda}^{(0)}(k+1)\\
&\boldsymbol{x}^{(0)}(k) = \boldsymbol{\phi}(k), \quad k \in I_0\\
&\boldsymbol{\lambda}^{(0)}(\infty) = 0
\end{aligned} \tag{13}$$

and

$$\begin{aligned}
&\boldsymbol{x}^{(i)}(k+1) = \boldsymbol{A}\boldsymbol{x}^{(i)}(k) - \boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{\lambda}^{(i)}(k+1) + \mathrm{i}\boldsymbol{A}_1\boldsymbol{x}^{(i-1)}(k-\tau)\\
&\boldsymbol{\lambda}^{(i)}(k) = \boldsymbol{C}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{C}\boldsymbol{x}^{(i)}(k) + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\lambda}^{(i)}(k+1) + \mathrm{i}\boldsymbol{A}_1^T\boldsymbol{\lambda}^{(i-1)}(k+\tau+1)\\
&\boldsymbol{x}^{(i)}(k) = 0, \quad k \in I_0\\
&\boldsymbol{\lambda}^{(i)}(\infty) = 0\\
&i = 1,2,\ldots
\end{aligned} \tag{14}$$

Substituting (11) into (10), we obtain

$$\boldsymbol{u}^{(i)}(k) = -\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{\lambda}^{(i)}(k+1), \quad i = 0,1,2,\ldots. \tag{15}$$

Thus, TPBV problem (7) is transformed into a series of TPBV problems in (13) and (14).

Remark 2. In the TPBV problem sequence in (13) and (14), the ith step TPBV problem contains only time-advance and time-delay terms of $(i-1)$th step. Therefore, an iterative process can be used to solve the TPBV problem sequence.

3.3. Approximation process of OTC law

We next give the control law of the OTC problem. In order to prove the existence and uniqueness of the OTC law, we would like to introduce a lemma first.

Lemma 1 [11]. *The Stein equation with respect to* $\boldsymbol{X} \in \boldsymbol{R}^{n\times m}$ *in the form of*

$$\overline{\boldsymbol{B}}\boldsymbol{X}\overline{\boldsymbol{A}} - \boldsymbol{X} = \boldsymbol{\Gamma} \tag{16}$$

where $\overline{\boldsymbol{A}} \in \boldsymbol{C}^{m\times m}$, $\overline{\boldsymbol{B}} \in \boldsymbol{C}^{n\times n}$, $\boldsymbol{\Gamma} \in \boldsymbol{C}^{n\times m}$, *has a unique solution if and only if*

$$\lambda_i(\overline{\boldsymbol{A}})\lambda_j(\overline{\boldsymbol{B}}) \neq 1, \quad i = 1,2,\cdots,m; \quad j = 1,2,\cdots,n \tag{17}$$

Theorem 1. *Consider the OTC problem of discrete time-delay system* (1) *with infinite horizon cost functional* (4) *with respect to the disturbances defined by* (2) *and the desired output described by exosystem* (3)*. Assume that the following conditions hold*:

(i) $(\boldsymbol{A},\boldsymbol{B})$ *is completely controllable and* $(\boldsymbol{A},\boldsymbol{C})$ *is completely observable*;
(ii) *The exosystem described by* (3) *is asymptotically stable.*

The OTC law is existent and unique. The OTC law is expressed in the form of

$$\begin{aligned}
\boldsymbol{u}^*(k) = -\boldsymbol{S}\boldsymbol{B}^{\mathrm{T}}\Bigg[&\boldsymbol{P}\boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{P}\boldsymbol{A}_1\boldsymbol{x}(k-\tau) + \boldsymbol{P}_1\boldsymbol{F}\boldsymbol{z}(k)\\
&+(\boldsymbol{P}\boldsymbol{M}\boldsymbol{D} + \boldsymbol{P}_2\boldsymbol{G})\boldsymbol{v}(k) + \sum_{i=1}^{\infty}\frac{1}{i!}\boldsymbol{g}_i(k+1)\Bigg]
\end{aligned} \tag{18}$$

where $\mathrm{S} = (\boldsymbol{R} + \boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{B})^{-1}$; $\boldsymbol{P}$ *is the unique positive-definite solution of the Riccati equation*

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{T}\boldsymbol{A} - \boldsymbol{P} + \boldsymbol{Q} = 0 \tag{19}$$

where $\boldsymbol{T} = \boldsymbol{I} - \boldsymbol{B}\boldsymbol{S}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}$; $\boldsymbol{P}_1$ *is the unique solution of the Stein equation*

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{T}^{\mathrm{T}}\boldsymbol{P}_1\boldsymbol{F} - \boldsymbol{P}_1 = \boldsymbol{C}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{H} \tag{20}$$

$\boldsymbol{P}_2$ *is the unique solution of the Stein equation*

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{T}^{\mathrm{T}}\boldsymbol{P}_2\boldsymbol{G} - \boldsymbol{P}_2 = -\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{T}\boldsymbol{M}\boldsymbol{D} \tag{21}$$

and $\boldsymbol{g}_i(k)$ *represents adjoint vectors solved by means of the following equation*:

$$\begin{aligned}
\boldsymbol{g}_i(k) = \sum_{m=0}^{\infty}\left[\boldsymbol{A}^{\mathrm{T}}\boldsymbol{T}^{\mathrm{T}}\right]^m\big[&\boldsymbol{i}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{T}\boldsymbol{A}_1\boldsymbol{x}^{(i-1)}(k-\tau+m)\\
&+\boldsymbol{i}\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{\lambda}^{(i-1)}(k+1+\tau+m)\big]; \quad i = 1,2,\ldots
\end{aligned} \tag{22}$$

where

$$
\begin{aligned}
\boldsymbol{x}^{(0)}(k+1) &= \boldsymbol{TAx}^{(0)}(k) - \boldsymbol{BSB}^{\mathrm{T}}\boldsymbol{P}_1\boldsymbol{Fz}(k) \\
&\quad + \left[\boldsymbol{MD} - \boldsymbol{BSB}^{\mathrm{T}}(\boldsymbol{PMD} + \boldsymbol{P}_2\boldsymbol{G})\right]\boldsymbol{v}(k) \\
\boldsymbol{x}^{(0)}(k) &= \boldsymbol{\phi}(k), \quad k \in I_0 \\
\boldsymbol{x}^{(i)}(k+1) &= \boldsymbol{TAx}^{(i)}(k) - \boldsymbol{BSB}\mathrm{T}\boldsymbol{g}_i(k+1) \\
&\quad + \mathrm{i}\boldsymbol{TA}_1\boldsymbol{x}^{(i-1)}(k-\tau) \\
\boldsymbol{x}^{(i)}(k) &= 0, \quad k \in I_0
\end{aligned} \tag{23}
$$

and

$$
\begin{aligned}
\boldsymbol{\lambda}^{(0)}(k) &= \boldsymbol{Px}^{(0)}(k) + \boldsymbol{P}_1\boldsymbol{z}(k) + \boldsymbol{P}_2\boldsymbol{v}(k) \\
\boldsymbol{\lambda}^{(i)}(k) &= \boldsymbol{Px}^{(i)}(k) + \boldsymbol{g}_i(k), \quad i = 1, 2, \ldots
\end{aligned} \tag{24}
$$

Proof. Firstly we consider the solution of $\boldsymbol{u}^{(0)}(k)$. Let $\boldsymbol{\lambda}^{(0)}(k)$ be the first equation of (24). Substituting the first equation of (24) into (15), and considering (2) and (3), the 0th step optimal control law can be written as

$$
\begin{aligned}
\boldsymbol{u}^{(0)}(k) &= -\boldsymbol{SB}^{\mathrm{T}}\left[\boldsymbol{PAx}^{(0)}(k) + \boldsymbol{P}_1\boldsymbol{Fz}(k)\right. \\
&\quad \left. + (\boldsymbol{PMD} + \boldsymbol{P}_2\boldsymbol{G})\boldsymbol{v}(k)\right]
\end{aligned} \tag{25}
$$

From (13), (15) and (25), we get the first equation of (23). Substituting the first equation of (23) into (25), we have

$$
\begin{aligned}
\boldsymbol{\lambda}^{(0)}(k+1) &= \boldsymbol{PTAx}^{(0)}(k) + \boldsymbol{T}^{\mathrm{T}}\boldsymbol{P}_1\boldsymbol{Fz}(k) \\
&\quad + \boldsymbol{T}^{\mathrm{T}}(\boldsymbol{PMD} + \boldsymbol{P}_2\boldsymbol{G})\boldsymbol{v}(k)
\end{aligned} \tag{26}
$$

From (26) and the second equation of (13), we get

$$
\begin{aligned}
\boldsymbol{\lambda}^{(0)}(k) &= \left[\boldsymbol{C}^{\mathrm{T}}\boldsymbol{QC} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{PTA}\right]x^{(0)}(k) \\
&\quad + \left[\boldsymbol{A}^{\mathrm{T}}\boldsymbol{T}^{\mathrm{T}}\boldsymbol{P}_1\boldsymbol{F} - \boldsymbol{C}^{\mathrm{T}}\boldsymbol{QH}\right]\boldsymbol{z}(k) \\
&\quad + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{T}^{\mathrm{T}}(\boldsymbol{PMD} + \boldsymbol{P}_2\boldsymbol{G})\boldsymbol{v}(k)
\end{aligned} \tag{27}
$$

Compare the first equation of (24) with Eq. (27). Since the right-hand side terms of (24) and (27) hold for any $\boldsymbol{x}^{(0)}(k)$, $\boldsymbol{z}(k)$ and $\boldsymbol{v}(k)$, we can obtain the Riccati equation (19), the Stein equations (20) and (21), which yield the solutions for $\boldsymbol{P}(k)$, $\boldsymbol{P}_1(k)$ and $\boldsymbol{P}_2(k)$, respectively.

We next consider the solution of $\boldsymbol{u}^{(i)}(k)$ $(i = 1, 2, \cdots)$. Let $\boldsymbol{\lambda}^{(i)}(k)$ be the second equation of (24). Using the first equation of (14), (15) and the second equation of (24), we can get the ith step optimal control law

$$
\begin{aligned}
\boldsymbol{u}^{(i)}(k) &= -\boldsymbol{SB}^{\mathrm{T}}\left[\boldsymbol{PAx}^{(i)}(k) + \mathrm{i}\boldsymbol{PA}_1\boldsymbol{x}^{(i-1)}(k-\tau)\right. \\
&\quad \left. + \boldsymbol{g}_i(k+1)\right]
\end{aligned} \tag{28}
$$

Substituting (28) and (15) into (14), we obtain the third equation of (23). From the third equation of (23) and the second equation of (24), we obtain

$$
\begin{aligned}
\boldsymbol{\lambda}^{(i)}(k+1) &= \boldsymbol{PTAx}^{(i)}(k) + \boldsymbol{T}^{\mathrm{T}}\boldsymbol{g}_i(k+1) \\
&\quad + \mathrm{i}\boldsymbol{PTA}_1\boldsymbol{x}^{(i-1)}(k-\tau)
\end{aligned} \tag{29}
$$

Using the second equation of (24), (29), and the second equation of (14), we can get

$$
\begin{aligned}
\boldsymbol{\lambda}^{(i)}(k) &= \left[\boldsymbol{C}^{\mathrm{T}}\boldsymbol{QC} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{PTA}\right]\boldsymbol{x}^{(i)}(k) + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{T}^{\mathrm{T}}\boldsymbol{g}_i(k+1) \\
&\quad + \mathrm{i}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{PTA}_1\boldsymbol{x}^{(i-1)}(k-\tau) \\
&\quad + \mathrm{i}\boldsymbol{A}_1^{\mathrm{T}}\left[\boldsymbol{Px}^{(i-1)}(k+\tau+1) + \boldsymbol{g}_{i-1}(k+\tau+1)\right]
\end{aligned} \tag{30}
$$

Comparing the second equation of (24) with Eq. (30) and using (19), we can obtain the adjoint difference equation in the form of

$$
\begin{aligned}
\boldsymbol{g}_i(k) &= \boldsymbol{A}^{\mathrm{T}}\boldsymbol{T}^{\mathrm{T}}\boldsymbol{g}^{(i)}(k+1) + \mathrm{i}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{PTA}_1\boldsymbol{x}^{(i-1)}(k-\tau) \\
&\quad + \mathrm{i}\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{Px}^{(i-1)}(k+1+\tau) + \mathrm{i}\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{g}_{i-1}(k+1+\tau)
\end{aligned} \tag{31}
$$

The solution of adjoint difference Eq. (31) can be described as

$$
\begin{aligned}
\boldsymbol{g}_i(k) &= \lim_{m\to\infty}\left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{T}^{\mathrm{T}}\right)^m g_i(m) \\
&\quad + \sum_{m=0}^{\infty}\left\{\left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{T}^{\mathrm{T}}\right)^m\left[\mathrm{i}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{PTA}_1\boldsymbol{x}^{(i-1)}(k-\tau+m)\right.\right. \\
&\quad + \mathrm{i}\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{Px}^{(i-1)}(k+1+\tau+m) \\
&\quad \left.\left. + \mathrm{i}\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{g}_{i-1}(k+1+\tau+m)\right]\right\}
\end{aligned} \tag{32}
$$

Using the last equation of (14) and the second equation of (24), we get

$$
\boldsymbol{g}_i(\infty) = -\boldsymbol{Px}^{(i)}(\infty) \tag{33}
$$

Noting the fact that $\boldsymbol{x}^{(i)}(k)$ are bounded as k increases to infinity, we have

$$
\lim_{N\to\infty}\|\boldsymbol{g}_i(N)\| < \gamma \tag{34}
$$

where γ is a positive integer related to $v(\infty)$. According to the theory of optimal control, we have

$$
|\lambda_i(TA)| = \left|\lambda_i\left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{T}^{\mathrm{T}}\right)\right| < 1 \tag{35}
$$

Thus, we have $\lim_{m\to\infty}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{T}^{\mathrm{T}})^m = 0$. From (32) we can obtain the solution of $g_i(k)$ in (22). Substituting (25) and (28) into (12), we obtain the OTC law (18).

In order to prove the existence and uniqueness of the optimal control law, it is only required to prove the existence and uniqueness of the positive-definite matrix $\boldsymbol{P}$, the matrices $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$, respectively. The existence and uniqueness of the positive-definite matrix $\boldsymbol{P}$ follow from condition (i) of Theorem 1 that $(\boldsymbol{A}, \boldsymbol{B})$ is completely controllable and $(\boldsymbol{A}, \boldsymbol{C})$ is completely observable. Next, we will prove the existence and uniqueness of $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$, respectively. From (35), condition (ii) of Theorem 1, and the exosystem (2), we have

$$
\lambda_i\left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{T}^{\mathrm{T}}\right)\lambda_j(\boldsymbol{F}) \neq 1, \quad i = 1, 2, \cdots, n; \; j = 1, 2, \cdots, n \tag{36}
$$

and

$$
\lambda_i\left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{T}^{\mathrm{T}}\right)\lambda_j(\boldsymbol{G}) \neq 1, \quad i = 1, 2, \cdots, n; \quad j = 1, 2, \cdots, n \tag{37}
$$

According to Lemma 1 we know that $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ are existent and unique, which are the solutions of Stein equations (20) and (21), respectively. The proof is complete. □

4. Approximately optimal algorithm

The infinite series $\sum_{i=1}^{\infty}\frac{1}{i!}\boldsymbol{g}_i(k+1)$ in OTC law (18) is almost impossible to be solved precisely. In practice, a suboptimal control law is obtained by replacing ∞ with some positive integer $\boldsymbol{M}$. The Mth step suboptimal tracking control (SOTC) law can be written as

$$\boldsymbol{u}_M(k) = -\boldsymbol{SB}^{\mathrm{T}}\Big\{\boldsymbol{P}[\boldsymbol{Ax}(k)+\boldsymbol{A}_1\boldsymbol{x}(k-\tau)]+\boldsymbol{P}_1\boldsymbol{Fz}(k) + [\boldsymbol{PMD}+\boldsymbol{P}_2\boldsymbol{G}]\boldsymbol{v}(k)+\sum_{i=1}^{M}\frac{1}{\boldsymbol{i}!}\boldsymbol{g}_i(k+1)\Big\} \tag{38}$$

From (38), we can get an iteration formula for the SOTC law

$$\begin{aligned}\boldsymbol{u}_0(k) &= -\boldsymbol{SB}^{\mathrm{T}}[\boldsymbol{PAx}(k)+\boldsymbol{PA}_1\boldsymbol{x}(k-\tau) + \boldsymbol{P}_1\boldsymbol{Fz}(k)+(\boldsymbol{PMD}+\boldsymbol{P}_2\boldsymbol{G})\boldsymbol{v}(k)]\\ \boldsymbol{u}_i(k) &= \boldsymbol{u}_{i-1}(k)-\frac{1}{i!}\boldsymbol{SB}^{\mathrm{T}}\boldsymbol{g}_i(k+1)\\ i &= 1,2,\cdots,M\end{aligned} \tag{39}$$

We can select $\boldsymbol{M}$ according to the control precision of the performance index. Next, we will give a practical algorithm calculating the SOTC law in (39) by using the following steps:

Step 1: Obtain $\boldsymbol{P}$, $\boldsymbol{P}_1$, $\boldsymbol{P}_2$ and $\boldsymbol{u}_0(k)$ from (19)–(21) and the first equation of (39), respectively; choose a small enough number $\delta>0$; let $\boldsymbol{J}_{-1}=\infty$ and $i=1$.

Step 2: Get the closed loop system by substituting $\boldsymbol{u}_i(k)$ into system (1); obtain the desired output $\tilde{\boldsymbol{y}}(k)$ from (3), $\boldsymbol{e}(k)$ from (5), and J_0 from the equation

$$\boldsymbol{J}_i=\sum_{k=0}^{\infty}\left[\boldsymbol{e}^{\mathrm{T}}(k)\boldsymbol{Qe}(k)+\boldsymbol{u}_i^{\mathrm{T}}(k)\boldsymbol{Ru}_i(k)\right] \tag{40}$$

Step 3: If $|(\boldsymbol{J}_i-\boldsymbol{J}_{i-1})/\boldsymbol{J}_M|<\delta$, let $M=i$, output the SOTC law $\boldsymbol{u}_M(k)$ and stop.

Step 4: Otherwise letting $\boldsymbol{i}=\boldsymbol{i}+1$; obtain $\boldsymbol{g}_i(k)$ from (22) and get $\boldsymbol{u}_i(k)$ by substituting $\boldsymbol{g}_i(k)$ into (39), go to step 2.

5. Physically realizable problem of the OTC law

Note that the OTC law in (18) contains the state vectors $\boldsymbol{v}$ and $\boldsymbol{z}$ of exosystems (2) and (3), which are physically unrealizable. In practical engineering, we can reconstruct state vectors $\boldsymbol{v}$ and $\boldsymbol{z}$ by using observers. In this section, we consider the construction problem of the reduced-order disturbance observer and reference input observer, respectively.

It is well known that for the full rank matrix $\boldsymbol{D}$ in (2), a constant matrix $\boldsymbol{L}\in\boldsymbol{R}^{(q-p)\times q}$ must exist such that the matrix $[\boldsymbol{D}^{\mathrm{T}}\boldsymbol{L}^{\mathrm{T}}]\in\boldsymbol{R}^{q\times q}$ is nonsingular. Let

$$\overline{\boldsymbol{T}}=\begin{bmatrix}\boldsymbol{D}\\ \boldsymbol{L}\end{bmatrix}^{-1}=[\,\overline{\boldsymbol{T}}_1\quad\overline{\boldsymbol{T}}_2\,],\quad \overline{\boldsymbol{T}}^{-1}\boldsymbol{G}\overline{\boldsymbol{T}}=\begin{bmatrix}\boldsymbol{G}_1 & \boldsymbol{G}_{12}\\ \boldsymbol{G}_{21} & \boldsymbol{G}_2\end{bmatrix} \tag{41}$$

where $\overline{\boldsymbol{T}}_1\in\boldsymbol{R}^{q\times p}$, $\overline{\boldsymbol{T}}_2\in\boldsymbol{R}^{q\times(q-p)}$, $\boldsymbol{G}_1\in\boldsymbol{R}^{p\times p}$, $\boldsymbol{G}_{12}\in\boldsymbol{R}^{p\times(q-p)}$, $\boldsymbol{G}_{21}\in\boldsymbol{R}^{(q-p)\times p}$, and $\boldsymbol{G}_2\in\boldsymbol{R}^{(q-p)\times(q-p)}$. In order to construct a disturbance observer, we make an equivalent linear transformation $\boldsymbol{v}=\overline{\boldsymbol{T}}\bar{\boldsymbol{v}}$. Denote that $\bar{\boldsymbol{v}}^{\mathrm{T}}=[\,\bar{\boldsymbol{v}}_1^{\mathrm{T}}\quad\bar{\boldsymbol{v}}_2^{\mathrm{T}}]$, where $\bar{\boldsymbol{v}}_1\in\boldsymbol{R}^p$, $\bar{\boldsymbol{v}}_2\in\boldsymbol{R}^{(q-p)}$. An equivalent system of exosystem (2) is obtained as follows:

$$\begin{aligned}\bar{\boldsymbol{v}}_1(k+1)&=\boldsymbol{G}_1\bar{\boldsymbol{v}}_1(k)+\boldsymbol{G}_{12}\bar{\boldsymbol{v}}_2(k)\\ \bar{\boldsymbol{v}}_2(k+1)&=\boldsymbol{G}_{21}\bar{\boldsymbol{v}}_1(k)+G_2\bar{\boldsymbol{v}}_2(k)\\ \boldsymbol{w}(k)&=\bar{\boldsymbol{v}}_1(k)\end{aligned} \tag{42}$$

Note that $\bar{\boldsymbol{v}}_1$ is just the output $\boldsymbol{w}$ of exosystem (2). We only need to construct a reduced-order observer with respect to $\bar{\boldsymbol{v}}_2$. Noting that $\boldsymbol{D}\overline{\boldsymbol{T}}=[\,I_m\quad 0\,]$ and $(\boldsymbol{G},\boldsymbol{D})$ is completely observable, we can obviously see that $(\boldsymbol{G}_2,\boldsymbol{G}_{12})$ is also completely observable. Construct the reduced-order disturbance observer as follows:

$$\begin{aligned}\boldsymbol{\sigma}(k+1)&=\widehat{\boldsymbol{G}}\boldsymbol{\sigma}(k)+\widehat{\boldsymbol{D}}\boldsymbol{w}(k)\\ \hat{\boldsymbol{v}}_2(k)&=\boldsymbol{\sigma}(k)+\boldsymbol{Kw}(k)\end{aligned} \tag{43}$$

where $\boldsymbol{\sigma}\in\boldsymbol{R}^{(q-p)}$ is a reconstructed vector; $\widehat{\boldsymbol{G}}=\boldsymbol{G}_2-\boldsymbol{KG}_{12}$, $\widehat{\boldsymbol{D}}=\boldsymbol{G}_2\boldsymbol{K}-\boldsymbol{KG}_{12}\boldsymbol{K}+\boldsymbol{G}_{21}-\boldsymbol{KG}_1$; $\hat{\boldsymbol{v}}_2$ is the observing value of $\bar{\boldsymbol{v}}_2$; K is a gain matrix to be found. In order to guarantee the speediness and nicety of observer in (43), we can select matrix K such that all the eigenvalues of matrix $\boldsymbol{G}_2-\boldsymbol{KG}_{12}$ are assigned to appointed places. From (2), (41) and (43), we can get the observing value of $\boldsymbol{v}(k)$ as

$$\hat{\boldsymbol{v}}(k)=\overline{\boldsymbol{T}}_2\boldsymbol{\sigma}(k)+(\overline{\boldsymbol{T}}_1+\overline{\boldsymbol{T}}_2\boldsymbol{K})\boldsymbol{w}(k) \tag{44}$$

Similarly, we can construct the reduced-order reference input observer as

$$\begin{aligned}\boldsymbol{\eta}(k+1)&=\widehat{\boldsymbol{F}}\boldsymbol{\eta}(k)+\widehat{\boldsymbol{H}}\boldsymbol{y}_s(k)\\ \hat{\boldsymbol{z}}_2(k)&=\boldsymbol{\eta}(k)+\widehat{\boldsymbol{K}}\boldsymbol{y}_s(k)\end{aligned} \tag{45}$$

The observing value of $\boldsymbol{z}(k)$ is as follows:

$$\hat{\boldsymbol{z}}(k)=\widehat{\boldsymbol{T}}_2\boldsymbol{\eta}(k)+(\widehat{\boldsymbol{T}}_1+\widehat{\boldsymbol{T}}_2\widehat{\boldsymbol{K}})y_s(k) \tag{46}$$

where $\boldsymbol{\eta}\in\boldsymbol{R}^{(l-m)}$ is a reconstructed vector; $\hat{\boldsymbol{z}}_2$ is the observing value of $\boldsymbol{z}_2$;

$$\widehat{\boldsymbol{F}}=\boldsymbol{F}_2-\widehat{\boldsymbol{K}}\boldsymbol{F}_{12},\quad \widehat{\boldsymbol{H}}=\boldsymbol{F}_2\widehat{\boldsymbol{K}}-\widehat{\boldsymbol{K}}\boldsymbol{F}_{12}\widehat{\boldsymbol{K}}+\boldsymbol{F}_{21}-\widehat{\boldsymbol{K}}\boldsymbol{F}_1$$

$$\widehat{\boldsymbol{T}}=\begin{bmatrix}\boldsymbol{H}\\ \boldsymbol{M}\end{bmatrix}^{-1}=[\,\widehat{\boldsymbol{T}}_1\quad\widehat{\boldsymbol{T}}_2\,],\quad \widehat{\boldsymbol{T}}^{-1}\boldsymbol{F}\widehat{\boldsymbol{T}}=\begin{bmatrix}\boldsymbol{F}_1 & \boldsymbol{F}_{12}\\ \boldsymbol{F}_{21} & \boldsymbol{F}_2\end{bmatrix}$$

where $\widehat{\boldsymbol{K}}$ is a gain matrix to be found. In order to guarantee the speediness and nicety of observer in (45), we can select matrix $\widehat{\boldsymbol{K}}$ such that all the eigenvalues of matrix $\boldsymbol{F}_2-\widehat{\boldsymbol{K}}\boldsymbol{F}_{12}$ are assigned to appointed places.

Through the reconstructions of $\boldsymbol{v}(k)$ and $\boldsymbol{z}(k)$, we can obtain a dynamic OTC law in the form of

$$\boldsymbol{u}(k) = -\boldsymbol{S}\boldsymbol{B}^{\mathrm{T}}\Bigg\{\boldsymbol{P}\boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{P}\boldsymbol{A}_1\boldsymbol{x}(k-\tau) + \boldsymbol{P}_1\boldsymbol{F}\left[\widehat{\boldsymbol{T}}_2\boldsymbol{\eta}(k) + (\widehat{\boldsymbol{T}}_1 + \widehat{\boldsymbol{T}}_2\widehat{\boldsymbol{K}})\boldsymbol{y}_s(k)\right] + (\boldsymbol{PMD} + \boldsymbol{P}_2\boldsymbol{G})\left[\overline{\boldsymbol{T}}_2\boldsymbol{\sigma}(k) + (\overline{\boldsymbol{T}}_1 + \overline{\boldsymbol{T}}_2\boldsymbol{K})\boldsymbol{w}(k)\right] + \sum_{i=1}^{\infty}\frac{\mathbf{1}}{\mathbf{i}!}\boldsymbol{g}_i(k+1)\Bigg\} \tag{47}$$

In practical engineering, a suboptimal control law is obtained by replacing ∞ with some positive integer $\boldsymbol{M}$. Then the $\boldsymbol{M}$th dynamic SOTC law may be written as

$$\boldsymbol{u}_M(k) = -\boldsymbol{S}\boldsymbol{B}^{\mathrm{T}}\Bigg\{\boldsymbol{P}\boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{P}\boldsymbol{A}_1\boldsymbol{x}(k-\tau) + \boldsymbol{P}_1\boldsymbol{F}\left[\widehat{\boldsymbol{T}}_2\boldsymbol{\eta}(k) + \left(\widehat{\boldsymbol{T}}_1 + \widehat{\boldsymbol{T}}_2\widehat{\boldsymbol{K}}\right)\boldsymbol{y}_s(k)\right] + (\boldsymbol{PMD} + \boldsymbol{P}_2\boldsymbol{G})\left[\overline{\boldsymbol{T}}_2\boldsymbol{\sigma}(k) + (\overline{\boldsymbol{T}}_1 + \overline{\boldsymbol{T}}_2\boldsymbol{K})\boldsymbol{w}(k)\right] + \sum_{i=1}^{M}\frac{1}{\boldsymbol{i}!}\boldsymbol{g}_i(k+1)\Bigg\} \tag{48}$$

6. Simulations

Consider the 2nd-order discrete system described by (1) and (2), where

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0.1 \\ -0.2 & 0.9 \end{bmatrix}, \quad \boldsymbol{A}_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & -0.1 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix},$$
$$\boldsymbol{M} = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \quad \boldsymbol{C} = [0 \quad 20]$$

The disturbance vector is described by exosystem (2), where

$$\boldsymbol{G} = \begin{bmatrix} 0.95 & 0.18 \\ -0.18 & 0.9 \end{bmatrix}, \quad \boldsymbol{D} = [1 \quad 0.1], \quad v(0) = [0 \quad 1]^{\mathrm{T}}$$

The initial condition of the system is $\boldsymbol{\varphi}(k) = [0 \quad 0.5]^{\mathrm{T}}, k = -8, -7, \ldots, -1, 0$. The desired output can be described by (3), where

$$F = \begin{bmatrix} 0.9 & 0.1 \\ -0.2 & 0.8 \end{bmatrix}, \quad \boldsymbol{H} = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}^{\mathrm{T}}, \quad \boldsymbol{z}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

We choose quadratic performance index (4). Let the weighting matrices in (4) be $\boldsymbol{Q} = 8$ and $\boldsymbol{R} = 1$.

The simulation curves of the output error, state variable, and control law with the two methods are presented in Figs. 1–3, respectively. The performance index values J_i at different iteration times are listed in Table 1.

From Table 1, we can see that $J_i > J_{i+1} (i = 1,2,\ldots)$. That is, the performance index values decrease as iteration time increases. By expanding the system's dimension, we obtain the optimal performance index $J^* = 1.957$. Hence, $J_{18} = 1.959$ is very near the optimal performance index J^*. From the simulation results, we can conclude that the proposed algorithm is effective under different time-delays, especially for systems with long time-delay.

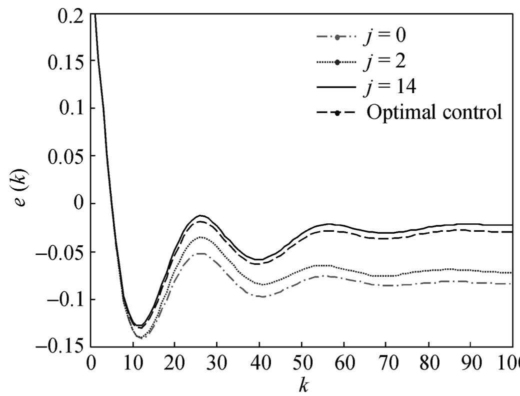

Fig. 1. Simulation curves of output error $\boldsymbol{e}(k)$.

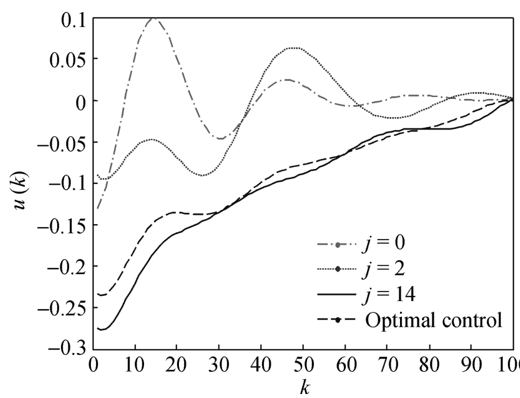

Fig. 2. Simulation curves of control law $\boldsymbol{u}(k)$.

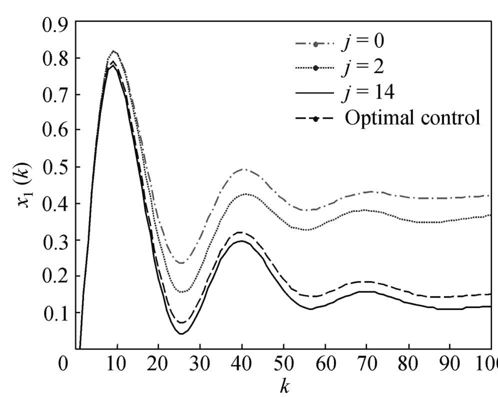

Fig. 3. Simulation curves of state variable $\boldsymbol{x}_1(k)$.

Table 1
Performance index values J_i

i	0	1	2	3
$\boldsymbol{J}_i$	3.284	2.869	2.622	2.372
i	6	10	15	18
$\boldsymbol{J}_i$	2.084	2.007	1.966	1.959

7. Conclusions

By introducing a sensitivity parameter, the original OTC problem has been transformed into a series of TPBV problems without time-advance and time-delay terms to avoid solving the TPBV problem with both time-advance and time-delay terms. An approximate regulator for the OTC problem of discrete time-delay systems affected by exogenous disturbances with a quadratic performance index has been presented. The simulation results have shown that the obtained regulator is easy to implement and robust with respect to additive persistent disturbances.

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